Mortar-Based Contact Formulation for Large Deformations

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The discontinuities due to the discretization lead to some challenges. First, the normal direction of the contact surface is not steady because the discrete surface is only C_0 continuous. One might smooth the normal vector field. Second, the question of contact enforcement has to be cleared. Contact forces can be modeled with either a Lagrange multiplier method or a penalty formulation to prevent penetration. Third, there must be developed a integration scheme which is able to handle the non steady boundary. Last, there is a strong discontinuity in measuring the penetration, where different criteria for enabling or disabling contact can be found (active set strategy). In this work different approaches to solve this tasks are presented and brought into context.

1 Mortar Finite Element Method

The mortar finite element method describes the coupling of non matching discretization of two subdomains. This coupling is realized by enforcing a relation between the two domains with the help of Lagrange multipliers. It is very characteristic that the geometric and the kinetic conditions are fulfilled in weak form. This method for coupling subdomains can also be applied on contact problems. The treatment of frictionless contact and solutions for the discretization issues are topic of this work.

2 Contact Formulation

Virtual work of contact forces We define a contact boundary Γ_C with contact force density $t_N^{(1)}$ on the non-mortar surface and $t_N^{(2)}$ on the mortar surface and the normal vector $\mathbf{n}$. It is more or less arbitrary which subdomain is treated as mortar or non-mortar domain. The problem of defining an appropriate normal vector is discussed in 3. The virtual work of contact forces $\delta\Pi_C(\mathbf{u},\delta\mathbf{u}) = -\sum_{i=1}^{2}\int_{\Gamma_C^{(i)}} t_N^{(i)}\ \mathbf{n}\cdot\delta\mathbf{u}^{(i)}\ \mathrm{d}\Gamma$ corresponds to the Neumann boundaries. For contact surfaces the equilibrium $t_N^{(1)}\mathrm{d}\gamma_C^{(1)} = -t_N^{(2)}\mathrm{d}\gamma_C^{(2)}$ and $\gamma_C^{(1)} = \gamma_C^{(2)} = \gamma_C$ in normal direction has to be fulfilled. If we use this equilibrium and focus on non-mortar force density $t_N^{(1)}$ we can formulate the virtual work of contact forces as shown in (1).

$$\delta\Pi_C(\mathbf{u},\delta\mathbf{u}) = -\int_{\gamma_C} t_N^{(1)}\ \mathbf{n}\cdot\left(\delta\mathbf{u}^{(1)} - \delta\mathbf{u}^{(2)}\right)\ \mathrm{d}\gamma_C \tag{1}$$

Karush-Kuhn-Tucker condition Beside the virtual work of contact forces we need Karush-Kuhn-Tucker conditions. These describe the main properties of mechanical contact and can be written as $g(\mathbf{X},t) \geq 0$, $t_N^{(1)} \leq 0$ and $t_N^{(1)} g(\mathbf{X},t) = 0$. Once again the continuous and strong condition $g(\mathbf{X},t) \geq 0$ can be written in a weak formulation (2).

$$0 \leq \int_{\gamma_C^{(i)}} \delta t_N^{(1)}\ g(\mathbf{X},t)\mathrm{d}\gamma_C = \int_{\gamma_C^{(i)}} \delta t_N^{(1)}\ \mathbf{n}\cdot\left(\mathbf{x}^{(2)} - \mathbf{x}^{(1)}\right)\mathrm{d}\gamma_C \tag{2}$$

It is also possible to retrieve the two conditions (1) and (2) by defining a variation of the contact potential $\Pi_C(\mathbf{u}) = \int_{\Gamma_C} t_N^{(1)}\ g(\mathbf{X},t)\mathrm{d}\Gamma$. [1]

Contact Enforcement What is left is the contact enforcement (which means the definition of $t_N^{(1)}$). There is the possibility to use a penalty method with $t_N^{(1)} = \varepsilon g(\mathbf{X},t)$ (and the corresponding base potential $\Pi_C = -\frac{1}{2}\int_{\Gamma_C}\varepsilon\, g(\mathbf{X},t)^2\mathrm{d}\Gamma$). The more common way (as mainly proposed for the mortar method) is the Lagrange method with $t_N^{(1)} = -\lambda_N$ (base potential $\Pi_C = -\int_{\Gamma_C}\lambda_N\, g(\mathbf{X},t)\mathrm{d}\Gamma$). Naturally, one may also use the augmented Lagrange method as some kind of method in between. By insertion of this definitions for $t_N^{(1)}$ into (1) and (2) the base equations for the finite element method are obtained.

3 Non Steady Normal Field

We have to decide which normal vector to use. We can either use the non-mortar normal $\mathbf{n}^{(1)}$ (Fig. 1) or the mortar side normal $\mathbf{n}^{(2)}$ (Fig. 2). The latter is more common because resulting in more compact expressions [1]. On the other hand the

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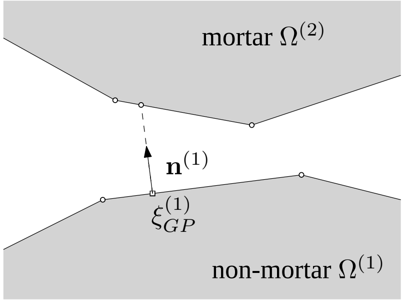

Fig. 1 non-mortar normal

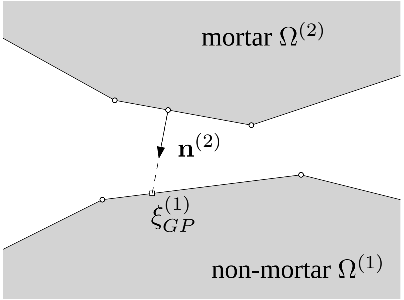

Fig. 2 mortar normal

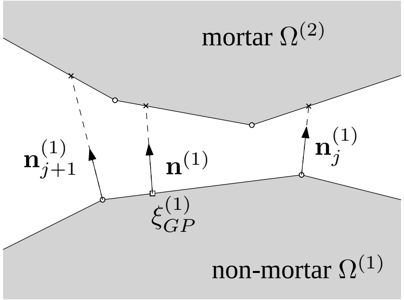

Fig. 3 non-mortar side averaged normal

non steady normal field may be smoothed by averaging the 2 normal vectors defined for each end node of the non-mortar edges (Fig. 3) [2]. This can be achieved for example by defining the arithmetic mean with $\mathbf{n}_j^{(1)} = \left(\mathbf{n}_{j1}^{(1)} + \mathbf{n}_{j2}^{(1)}\right) / \left(\|\mathbf{n}_{j1}^{(1)} + \mathbf{n}_{j2}^{(1)}\|\right)$ and $\mathbf{n}^{(1)} = \sum_{j=1}^{n_{(1)}} N_j(\xi^{(1)})\mathbf{n}_j^{(1)}$.

4 Numerical Integration Scheme

There are two possibilities to perform numerical integration over non-mortar edges. First, the "concentrated" integration (Fig. 4) where all integration relevant quantities are calculated in fixed integration points on the non-mortar edge [1]. Second, the more accurate numerical integration by splitting up the integration domain (Fig. 5). This is done by back projection of mortar nodes onto non-mortar edges with the previously defined normal vector [2].

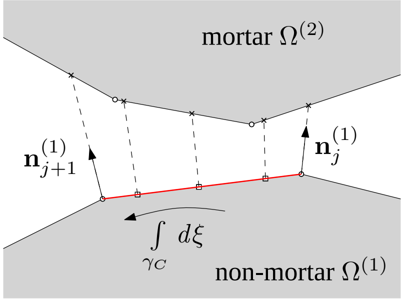

Fig. 4 concentrated integration scheme

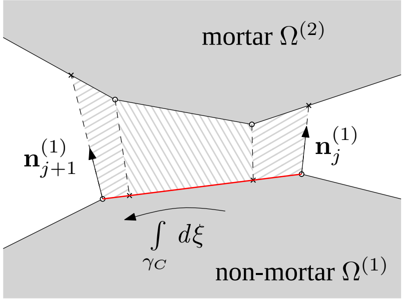

Fig. 5 integration by clipping

5 Active Set Strategy

We define an averaged nodal gap (based on (2)) with $\tilde{g}_j = \mathbf{n}^{(1)} \cdot \left[\int_{\gamma_c^h} \Phi_j N_l^{(2)} \mathrm{d}\gamma_C^h \mathbf{x}_l^{(2)} - \int_{\gamma_c^h} \Phi_j N_l^{(1)} \mathrm{d}\gamma_C^h \mathbf{x}_l^{(1)}\right]$. This averaged gap is used to formulate discrete conditions which are used to decide if nodes are in contact or not.

Fixed point Newton-Raphson In this case the active set is only updated convergence in the Newton-Raphson scheme is reached. We define an averaged nodal pressure like $\tilde{p}_j = -\left[\int_{\gamma_C{}^h} N_j^{(1)} \Phi_k \, \mathrm{d}\gamma_C^h \, (\lambda_N)_k\right] / \left[\int_{\gamma_C{}^h} N_j^{(1)} \, \mathrm{d}\gamma_C^h\right]$. For active nodes check if $\tilde{p}_j < 0$ and deactivate node if true. For inactive nodes check if $\tilde{g}_j < 0$ and activate node if true.

Semi-smooth Newton Raphson Here the active set problem and the displacement problem are consistently solve inside the Newton-Raphson scheme [3,4]. Therefore we define $C_j(\mathbf{t}_j, \mathbf{d}) = (z_n)_j - \max(0, (z_n)_j - c_n \, \tilde{g}_j)$ which continuously fulfills all Karush-Kuhn-Tucker conditions defined if $C_j(\mathbf{t}_j, \mathbf{d}) = 0$. In each iteration we activate a node if $(z_n)_j - c_n \, \tilde{g}_j > 0$ and deactivate a node if $(z_n)_j - c_n \, \tilde{g}_j \leq 0$.

Acknowledgements The author would like to thank Prof. Christian C. Celigoj, Graz University of Technology, Austria.

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